XLVII. *Of Logarithms, by the late* William Jones, *Esq; F. R. S. Communicated by* John Robertſon, *Lib. R. S.*

Read Dec. 5, 1771.

THE following paper on the nature and conſtruction of Logarithms, was communicated to me many years ſince, by that eminent Mathematician the late William Jones, Eſq. The familiar manner in which he explains their nature, and the great art with which he obtains the modes of computation, not being exceeded, if equaled, by any writer on this ſubject, may claim a place in the Philoſophical Tranſactions, to be preſerved among the multitude of excellent papers, of which that moſt invaluable work is a ſafe repoſitory.

OF LOGARITHMS.

1. Any number may be expreſſed by ſome ſingle power of the ſame radical number.

For every number whatever is placed ſomewhere in a ſcale of the ſeveral powers of ſome radical number r, whoſe indices are $m-1$, $m-2$, $m-3$, &c. where not only the numbers r^m, r^{m-1}, r^{m-2}, &c. are expreſſed; but alſo any intermediate number x is repreſented by r, with a proper index z.

 The

The index z is called the Logarithm of the number x.

2. Hence, to find the logarithm z of any number x, is only to find what power of the radical number r, in that ſcale, is equal to the number x; or to find the index z of the power, in the equation $x = r^z$.

3. The properties of logarithms are the ſame with the indices of powers; that is, the ſum or difference of the logarithms of two numbers, is the logarithm of the product or quotient of thoſe numbers.

And therefore, n times the logarithm of any number, is the logarithm of the nth power of that number.

4. The relation of any number x, and its logarithm z being given; To find the relation of their leaſt ſynchronal variation $\dot{x}$ and $\dot{z}$

Put $1+n$ for r, the radical number of any ſcale, and

$$q = \frac{n}{1+n}.$$

Let $a = q + \frac{1}{2}q^2 + \frac{1}{3}q^3 + \frac{1}{4}q^4$, &c.; $f = \frac{1}{a}$.

Then $f\dot{x} = x\dot{z}$ ſhews the relation required.

For $x = r^z = \overline{1+r}|^z$.

Now, let x and z flow ſo that x becomes $x+\dot{x}$, at the ſame time as z ſhall become $z+\dot{z}$,

$$\text{Then } x + \dot{x} = \overline{1+n}|^{z+\dot{z}} = \overline{1+n}|^{z} \times \overline{1+n}|^{\dot{z}}$$

$$= x \times \overline{1 + \dot{z}q + \tfrac{1}{2}\dot{z}q^2 + \tfrac{1}{3}\dot{z}q^3 + \tfrac{1}{4}\dot{z}q^4, \&c.}$$

$$\text{Therefore } \dot{x} = x\dot{z} \times \overline{q + \tfrac{1}{2}q^2 + \tfrac{1}{3}q^3 + \tfrac{1}{4}q^4, \&c.}$$

$$= x\dot{z}a = x\dot{z} \times \frac{1}{f}. \text{ Conſequently } f\dot{x} = x\dot{z}.$$

5. If

5. If $1+n=r=10$, as in the common logarithms of Briggs's form,

Then a will be found to be 2,302585092994, &c.

And $f=$ . . . . . 0,434294481903 25, &c.

If $a=1=f$, the form will be that of Napier's logarithms.

6. Let B, $\dot{\text{B}}$, be the logs of the numbers x, $\dot{x}$, in the form $f=\frac{1}{a}$,

And N, $\dot{\text{N}}$, the logs of the ſame numbers, in the form $\varphi=\frac{1}{\alpha}$.

Then $\text{B}\varphi=\text{N}f$; $\text{B}a=\text{N}\alpha$; $\text{B}\dot{\text{N}}=\text{N}\dot{\text{B}}$; $\dot{\text{B}}\varphi=\dot{\text{N}}f$.

For $\dot{\text{B}}:\dot{\text{N}}::(f\times\frac{\dot{x}}{x}:\varphi\times\frac{\dot{x}}{x}::)f:\varphi::\text{B}:\text{N}::\frac{1}{a}:\frac{1}{\alpha}::\dot{\text{B}}:\dot{\text{N}}$.

If $x=10$; $\text{B}=1$; $a=2,30258$, &c.; or $f=0,43429$, &c.; $\alpha=\varphi=1$.

Then $\text{N}=\text{B}\times\frac{a}{\alpha}=2,30258$, &c.

$$\dot{\text{N}}=\dot{\text{B}}\times\frac{\text{N}}{\text{B}}=2,30258,\ \text{\&c.}\times\dot{\text{B}}.$$

$$\dot{\text{B}}=\frac{f}{\varphi}\times\dot{\text{N}}=0,43429,\ \text{\&c.}\times\dot{\text{N}}.$$

7. Putting $x = q \pm v$; $N = \frac{v}{q}$.

Then $z =$ log. of x, or the log. of $q \pm v$, will be

$$= \pm\frac{v}{q} - \frac{v^2}{2q^2} \pm \frac{v^3}{3q^3} - \frac{v^4}{4q^4} \pm \frac{v^5}{5q^5} - \frac{v^6}{6q^6}, \&c. \times f;$$

$$= \pm N - \tfrac{1}{2}N^2 \pm \tfrac{1}{3}N^3 - \tfrac{1}{4}N^4 \pm \tfrac{1}{5}N^5 - \tfrac{1}{6}N^6, \&c. \times f.$$

For $\dot{z} = \dot{L}, x = \dot{L}, \overline{q \pm v} = f \times \frac{\dot{x}}{x} = f \times \frac{\pm \dot{v}}{q \pm v}$,

$$= \pm\frac{\dot{v}}{q} - \frac{\dot{v}v}{q^2} \pm \frac{\dot{v}v^2}{q^3} - \frac{\dot{v}v^3}{q^4} \pm \frac{\dot{v}v^4}{q^5} - \frac{\dot{v}v^5}{q^6}, \&c. \times f.$$

8. In three quantities p, q, r, increasing by equal differences, the logarithm of any one of them being given, the logarithms of the other two are also given.

For, let $v = q - p = r - q$; $N = \frac{v}{q} = \frac{q-p}{q} = \frac{r-q}{q}$;

P, Q, R, the logarithms of p, q, r.

I. $L, \frac{q}{p} = (L, \frac{q}{q-v} =) Q - P = f \times \overline{N + \tfrac{1}{2}N^2 + \tfrac{1}{3}N^3 + \tfrac{1}{4}N^4 + \tfrac{1}{5}N^5, \&c.} = fV.$

For $\dot{L}, \frac{q}{q-v} = f \times \frac{\dot{v}}{q-v}$.

II. $L, \frac{r}{q} = (L, \frac{q+v}{q} =) R - Q = f \times \overline{N - \tfrac{1}{2}N^2 + \tfrac{1}{3}N^3 - \tfrac{1}{4}N^4 + \tfrac{1}{4}N^5, \&c.} = fX.$

For $\dot{L}, \frac{q+v}{q} = f \times \frac{\dot{v}}{q+v}$.

III. $L, \frac{r}{p} = 2f \times \overline{V + X} = R - P = 2f \times \overline{N + \tfrac{1}{3}N^3 + \tfrac{1}{5}N^5 + \tfrac{1}{7}N^7 + \tfrac{1}{9}N^9, \&c.} = 2fZ.$

Where $N = (\frac{v}{q} =) \frac{r-p}{r+p}$.

Or, $L, \frac{r}{p} = L, \frac{q+v}{q-v} = R - P = 2fZ$.

For $\dot{L}, \frac{q+v}{q-v} = 2f \times \frac{q\dot{v}}{qq - vv}$.

9. Hence,

9, Hence, in two quantities, r the greater, p the less.

Putting $N = \frac{r-p}{r+p}$; $A = 2fN$; $B = AN^2$; $C = BN^2$; $D = CN^2$, &c.

And $S = A + \frac{1}{3}B + \frac{1}{5}C + \frac{1}{7}D$, &c.

Then L, $\frac{r}{p} = S$; Or $R - P = S$.

Or, putting $N = \frac{r-p}{r+p}$; $A = fN$, &c.

Then L, $\frac{r}{p} = 2S$.

Where $p = 1$; $N = \frac{r-1}{r+1}$; let $A = 2fN$, &c.

Then L, $r = S$.

Or, in this case, putting $N = \frac{r-1}{r+1} = A$; $B = AN^2$, &c.

Then L, $r = 2fS$.

Where $p = 1$, and $f = 1$; $N = \frac{r-1}{r+1}$; let $A = 2N$, &c.

Then L, $r = S$.

10. In three quantities p, q, r, increasing by equal differences, the logarithms of any two of them being given, the logarithm of the third is also given.

I. For L, $\frac{qq}{pr} = 2f \times \overline{V - X} = 2Q - \overline{P + R}$

$= 2f \times \frac{1}{2}N^2 + \frac{1}{4}N^4 + \frac{1}{6}N^6 + \frac{1}{8}N^8$, &c. $= 2fY$.

Where $N = \frac{r - p}{r + p}$.

Or L, $\frac{qq}{pr} =$ L, $\frac{qq}{qq - vv} = 2fY = 2Q - \overline{P + R}$.

Because $\dot{\text{L}}$, $\frac{qq}{qq - vv} = 2f \times \frac{\dot{v}v}{qq - vv}$.

II. Putting $N = \frac{qq - pr}{qq + pr} = \frac{vv}{qq + rp} =$ (where $v = 1$) $\frac{1}{qq + pr}$;

$A = fN$; $B = AN^2$, &c.

Then L, $\frac{qq}{pr} = 2S = 2Q - \overline{R + P}$; Or $Q - \frac{R + P}{2} = S$.

For since $vv = qq - pr = 1$; put qq for r; pr for q.

Then $r - p = qq - pr = vv = 1$; $r + p = qq + pr$.

III. Putting $N = \frac{v}{q} = A$, &c. $a = \frac{1}{2}$.

$b = \frac{1}{4} - \frac{1}{3}a$.

$c = \frac{1}{6} - \frac{1}{5}a - \frac{1}{3}b$,

$d = \frac{1}{8} - \frac{1}{7}a - \frac{1}{5}b - \frac{1}{3}c$.

$e = \frac{1}{10} - \frac{1}{9}a - \frac{1}{7}b - \frac{1}{5}c - \frac{1}{3}d$, &c.

And $M = aA + bB + cC + dD$, &c.; $\Sigma = \frac{1}{2}\overline{R + P}$; $\Delta = \frac{1}{2}\overline{R - P}$.

Then $Q = \Sigma + \Delta M$.

For $Q - P = fV = \alpha$; $R - P = 2fZ = 2\Delta$;

but $\left(\frac{\alpha}{\Delta} = 1 + M = \frac{Q - P}{\frac{1}{2}R - \frac{1}{2}P}\right.$; Therefore, &c.

11. Any

11. Any numbers p, q, r, &c. and as many ratios a, b, c, &c. composed of them, the difference of whose terms is 1; as also the logarithms A, B, C, &c. of those ratios, being given: To find the logarithms P, Q, R, &c. of those numbers, where the form is 1.

For instance, if $p=2$, $q=3$, $r=5$,

$$a=\left(\tfrac{9}{8}=\right)\frac{3^2}{2^3};\ b=\left(\tfrac{16}{15}=\right)\frac{2^4}{3.5};\ c=\left(\tfrac{25}{24}=\right)\frac{5^2}{3.2^3}.$$

Now, the logs A, B, C, of these ratios, a, b, c, being found, the log. of either 2, 3, 5, or of any number compounded of them, may be found directly, by making each successively equal to a^x, b^y, c^z.

Thus, for the log of $10=2.5$.

$$\text{Let } a^x b^y c^z = \frac{3^{2x}}{2^{3x}} \times \frac{2^{4y}}{3^y.5^y} \times \frac{5^{2z}}{3^z.2^{3z}}.$$

$$= 3^{2x}.2^{-3x} \times 2^{4y}.3^{-y}.5^{-y} \times 5^{2z}.3^{-z}.2^{3z} = 2.5.$$

Therefore $2^{4y-3x-3z-1} \times 3^{2x-y-z} \times 5^{2z-y-1} = 1$.

Consequently $4y-3x-3z-1=0$; $2x-y-z=0$; $2z-y-1=0$

Therefore $x = 10$; $y = 13$; $z = 7$;

and $a^{10} \times b^{13} \times c^7 = (2 \times 5 =)\ 10$.

Therefore $10A + 13B + 7C =$ log. of 10, to the form 1.

Or, since $a = \frac{3^2}{2^3}$; $b = \frac{2^4}{3.5}$; $c = \frac{5^2}{3.2^3}$.

Therefore $A = 2Q - 3P$; $B = 4P - Q - R$; $C = 2R - Q - 3P$.

Consequently
$$\left.\begin{aligned} P &= 3A + 4B + 2C = \text{log. of } 2 \\ Q &= 5A + 6B + 3C = \text{log. of } 3 \\ R &= 7A + 9B + 5C = \text{log. of } 5 \end{aligned}\right\} \text{to the form 1.}$$

Therefore $P + R = 10A + 13B + 7C =$ log. of $(2 \times 5 =)$ 10.

And fP, fQ, fR, are the logarithms of 2, 3, 3, respectively, in the scale of logarithms whose form is f.

 XLVIII. An